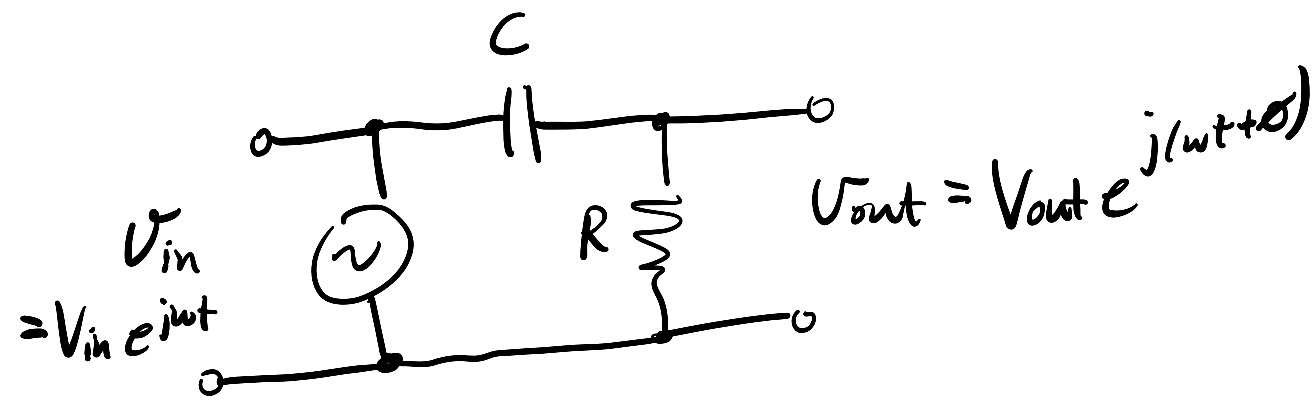
PHYS 331 - Sept. 18, 2023

Last Time:



When v_{out} is out of phase w.r.t. v_{in}, then v_{out} has real & imaginary components:

$X = V_{out} \cos\phi$ ← real component
$Y = V_{out} \sin\phi$ ← imaginary "

$$\tan\phi = \frac{Y}{X} \qquad V_{out} = \sqrt{X^2 + Y^2}$$

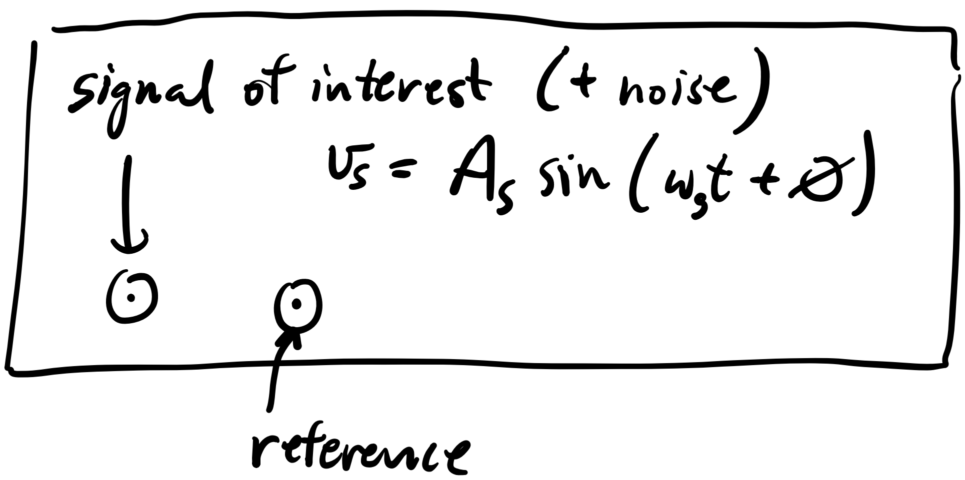
A lock-in detector is able to meas. the real & imag. components of small AC signals even when buried in noise.

If use lock-in to meas. X & Y, we can then find the amp. & phase of the signal

$$\theta = \tan^{-1}\left(\frac{Y}{X}\right) \qquad V_{out} = \sqrt{X^2+Y^2}$$

How does a lock-in detector meas. X & Y?

Lock-in detectors have two inputs



$$v_r = A_r \sin(\omega_r t + \theta)$$

We know everything about reference v_r.
We control A_r, ω_r, θ

Want to determine A_s & ϕ of v_s.

① Internally, lock-in detector multiplies the two inputs:

$$V_{out} = v_s v_r = A_s A_r \sin(\omega_s t + \phi) \sin(\omega_r t + \theta)$$

$$= \frac{A_s A_r}{2} \Big\{ \cos[(\omega_s t + \phi) - (\omega_r t + \theta)]$$

$$- \cos[(\omega_s t + \phi) + (\omega_r t + \theta)] \Big\}$$

low freq. term

$$v_{out} = \frac{A_s A_r}{2} \Big\{ \cos[(\omega_s - \omega_r) t + (\phi - \theta)]$$

high freq. term

$$- \cos[(\omega_s + \omega_r) t + (\phi + \theta)] \Big\}$$

② Next, the multiplied signal is passed through a low-pass filter to eliminate the high-freq. term.

$$V_{out} \rightarrow V_{out}' = \frac{A_s A_r}{2} \cos\left[(\omega_s - \omega_r)t + (\phi - \theta)\right]$$

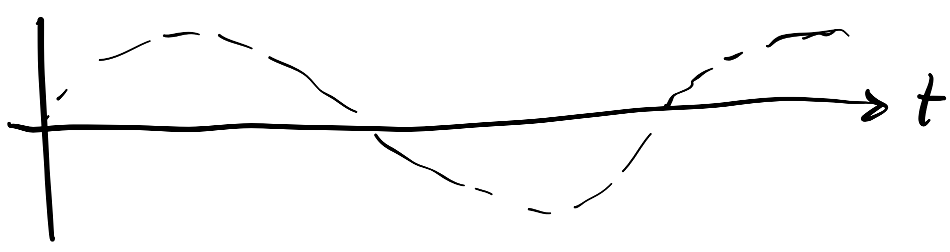
(after filter).

③ The filtered signal is then averaged over a long time.

time average

If $\omega_s \neq \omega_r$, then $\langle V_{out}' \rangle = 0$

b/c $\cos(()t)$ averages to zero (half pos. & half neg.)



Lock-in detector is blind to signal w/ freq. $\omega_s \neq \omega_r$.

If $\omega_s = \omega_r$, then

$$\left\langle V_{out}' \right\rangle_{\omega_s = \omega_r} = \frac{A_s A_r}{2} \cos(\phi - \theta) \quad \text{\textcircled{*}}$$

In ⊛, we still have two unkowns (A_s, ϕ) and only one eq'n.

④ To establish two eq'ns & two unkowns,

(a) set $\theta = 0$. Set ref. phase arbitrarily to zero.

$$\left\langle V_{out}' \right\rangle_{\omega_s = \omega_r} \Big|_{\theta = 0} = \frac{A_s A_r}{2} \cos\phi$$

$$= \frac{A_r}{2} \Big[\underbrace{A_s \cos\phi}_{X} \Big]$$

real component of V_s

$$\boxed{X = \frac{2}{A_r} \langle V_{out}' \rangle_{\omega_s=\omega_r} \Big|_{\theta=0}}$$ real component

(b) set $\theta = \frac{\pi}{2}$

$$\langle V_{out}' \rangle_{\omega_s=\omega_r} \Big|_{\theta=\frac{\pi}{2}} = \frac{A_s A_r}{2} \underbrace{\cos\left(\phi - \frac{\pi}{2}\right)}_{\sin\phi}$$

$$= \frac{A_r}{2} \underbrace{\left[A_s \sin\phi \right]}_{Y}$$

Y imaginary component of V_s

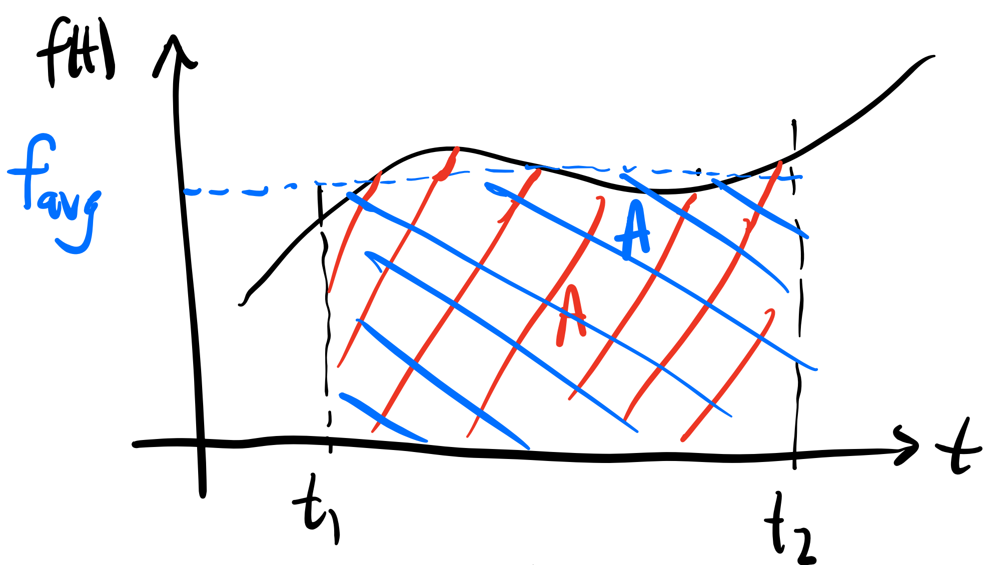
$$\boxed{Y = \frac{2}{A_r} \langle V_{out}' \rangle_{\omega_s=\omega_r} \Big|_{\theta=\frac{\pi}{2}}}$$

Lock-in detector is capable of meas. X & Y of out signal of interest v_s.
Can use these to then determine amp. & phase

$$A_s = \sqrt{X^2+Y^2} \qquad \phi = \tan^{-1}\left(\frac{Y}{X}\right)$$

Suppose that $\omega_s \approx \omega_r$ (but not identical).
How well does $\langle v_{out}' \rangle$ discriminate between freq. not equal to ω_r?

Aside



$$f_{avg} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} f(t)\,dt$$

set red & blue areas equal

The average of V_{out}' from $t=0$ to final time T.

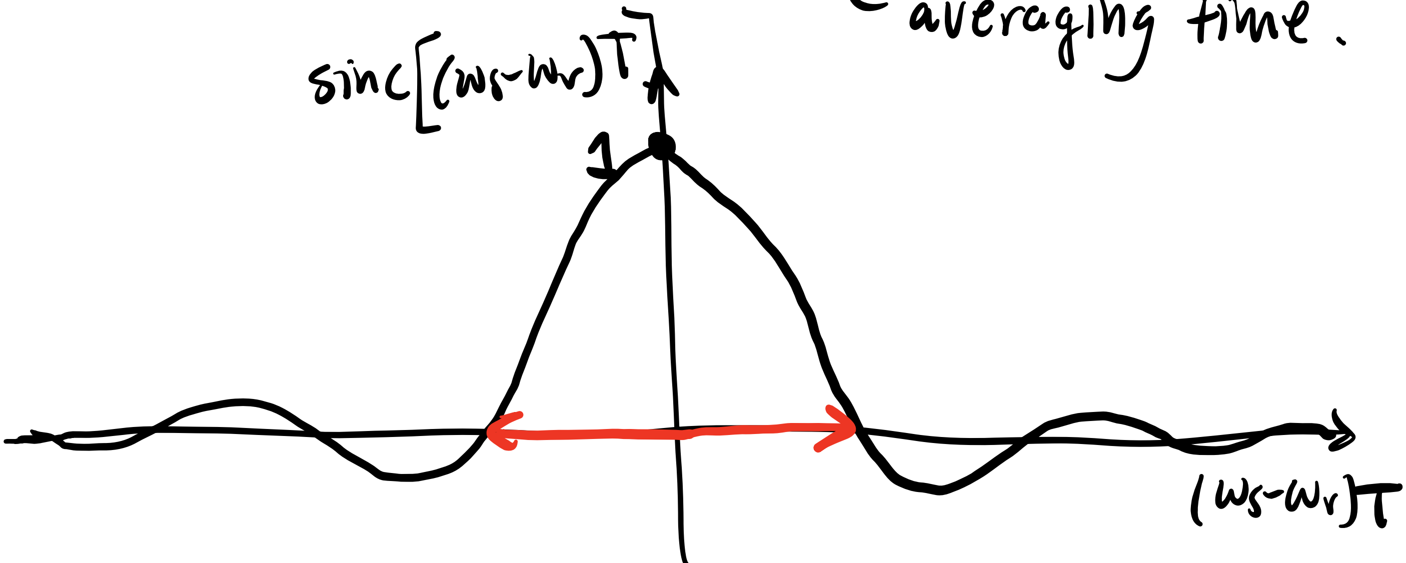
$$\langle V_{out}' \rangle \propto \frac{1}{T}\int_0^T \cos[(\omega_s-\omega_r)t]\,dt$$

$$\propto \frac{1}{T}\frac{1}{\omega_s-\omega_r}\sin[(\omega_s-\omega_r)t]\Big|_0^T$$

$$\propto \frac{1}{(\omega_s-\omega_r)T}\sin[(\omega_s-\omega_r)T]$$

$$\langle V_{out}' \rangle \propto \mathrm{sinc}[(\omega_s-\omega_r)T]$$

averaging time.



In order to characterize the width of $\langle V_{out}' \rangle$, find the values of $(\omega_s - \omega_r)T$ that produce the first zero crossings of $\mathrm{sinc}[(\omega_s - \omega_r)T]$.

The first zeros occur when

$$(\omega_s - \omega_r)T = \pm\pi$$

$$T_\pm = \frac{\pm\pi}{2\pi\underbrace{(f_s - f_r)}_{\Delta f}}$$

$$\Delta T = T_+ - T_-$$

$$\Delta T = \frac{\pi}{2\pi\Delta f} - \frac{-\pi}{2\pi\Delta f} = \frac{1}{\Delta f}$$

$$\boxed{\Delta T = \frac{1}{\Delta f}}$$

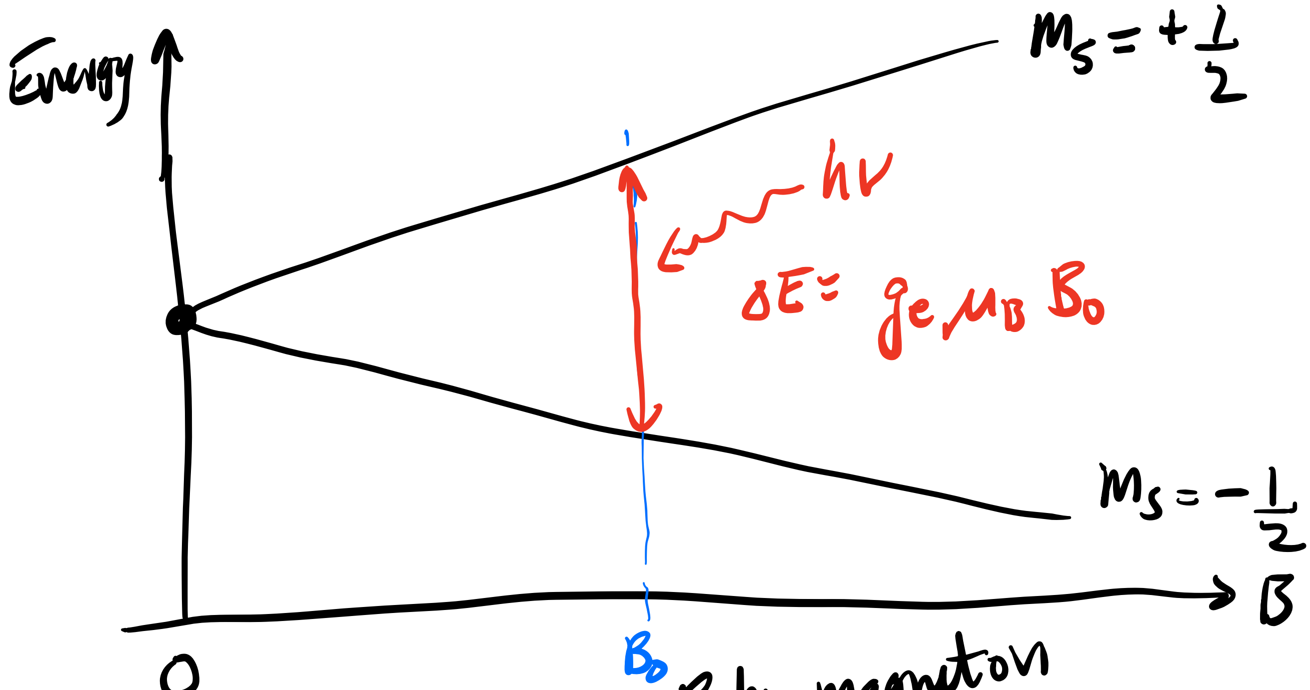
To suppress signal leakage from average, need averaging time to be much larger that $\frac{1}{\delta f}$.

In our example of an RC circuit, we supplied a sinusoidal input & detected a sinusoidal output.

Not all experiments can be driven by a pure sine wave. Consider, for example, Electron Spin Resonance (ESR).
ESR is used to study materials that have so-called unpaired electrons.
Consider a system of spin-$\frac{1}{2}$ electrons with spin angular momentum $m_s = \pm\frac{1}{2}$

In an external magnetic field applied along z-axis, the energy levels of $m_s = -\frac{1}{2}$

and $m_s = +\frac{1}{2}$ states split.

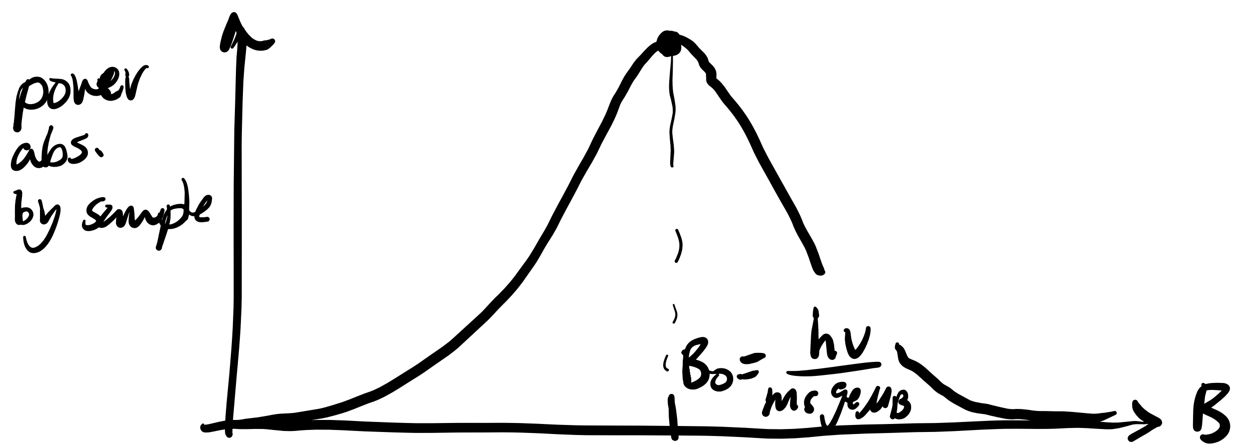


Bohr magneton

$$E = m_s g_e \mu_B B$$ magnetic field.

Landé g-factor

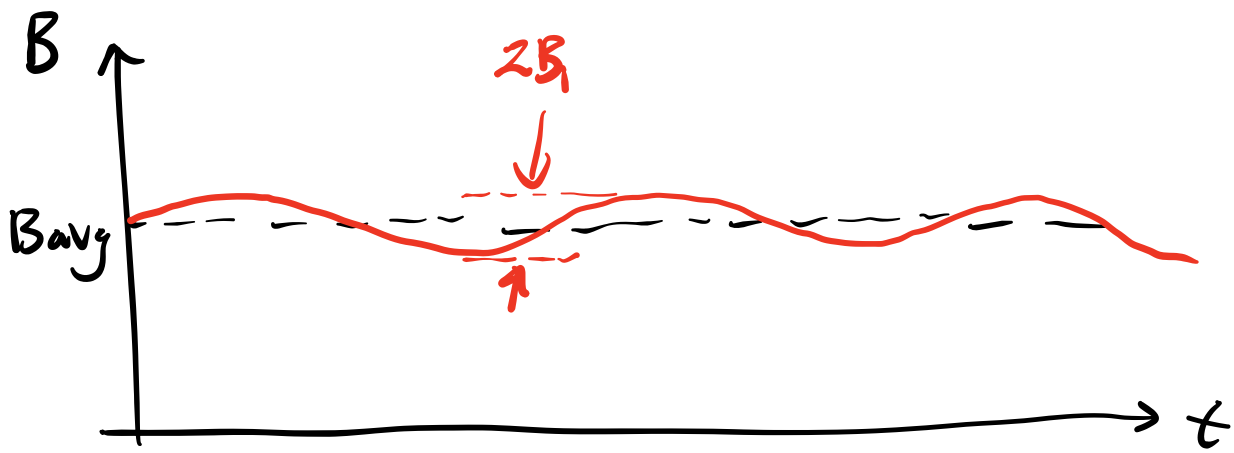
If external field is set to B_0, then a sample irradiated w/ EM waves w/ energy $h\nu$ can cause transitions between the two states.



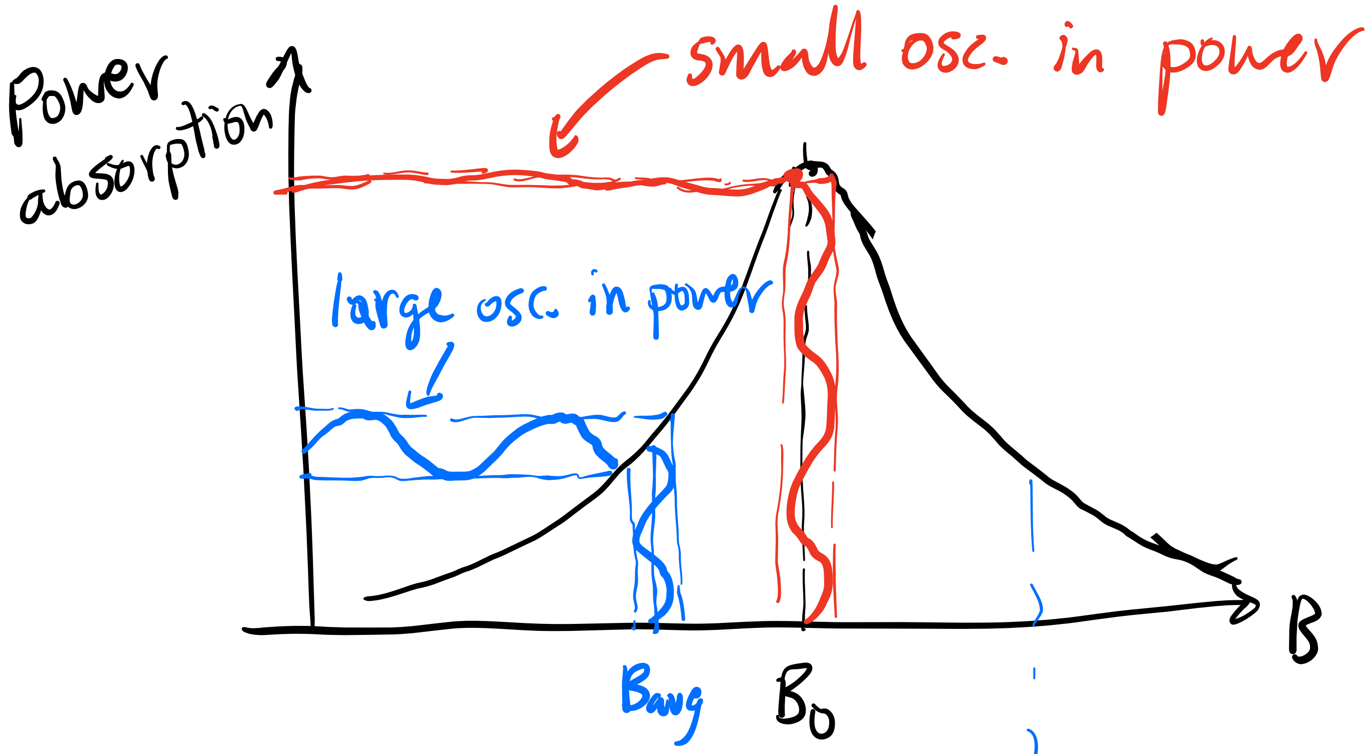
Challenge: The absorption peak can be very weak.

Soln: Use lock-in detection.
We need to introduce a sinusoidal signal to the experiment.

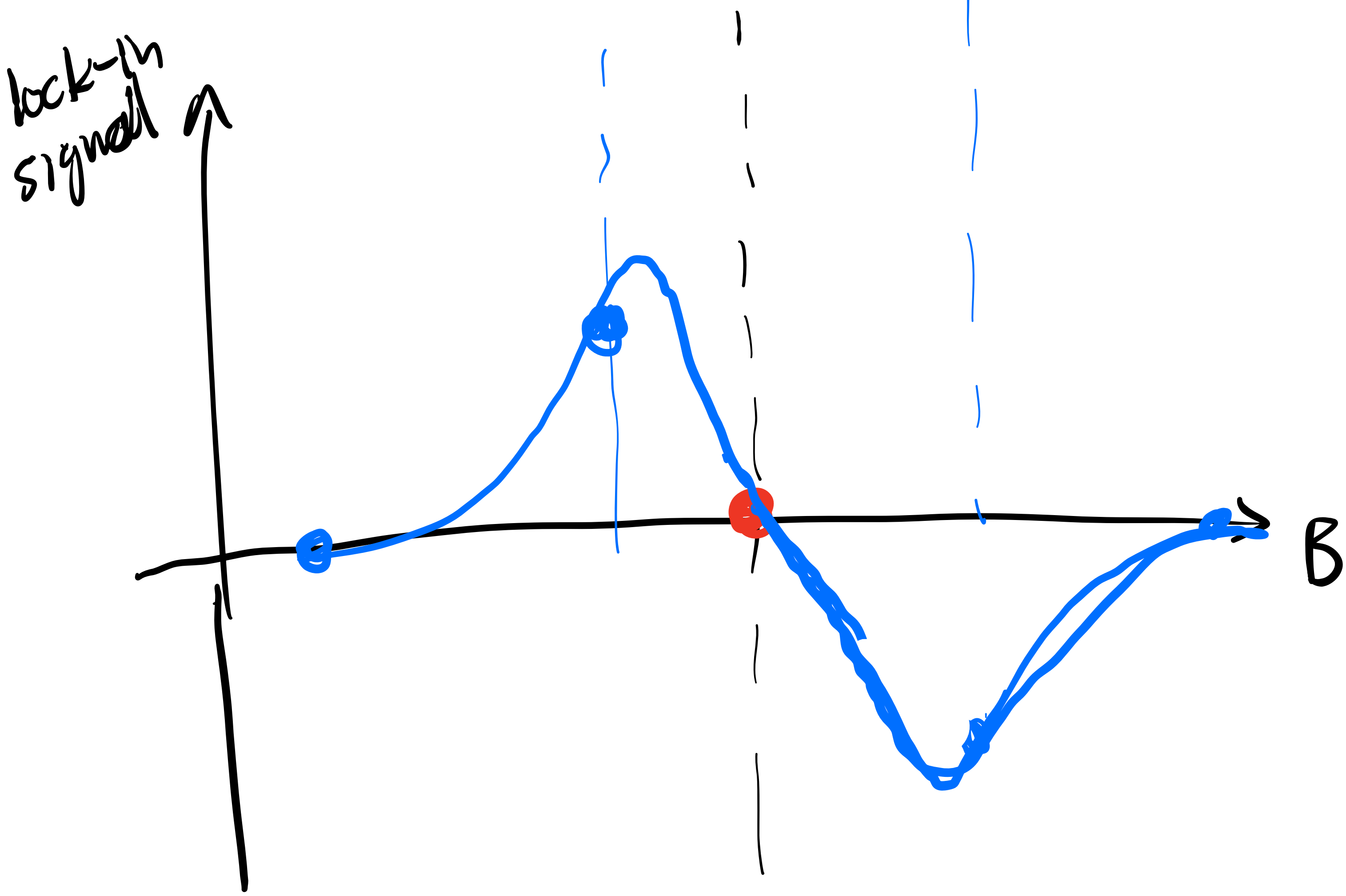
Idea is to add a <u>small</u> AC signal on top of avg. or DC magnetic field.



$$B = B_{avg} + \underbrace{B_1 \sin(\omega t)}_{\text{ref. signal.}}$$



Osc. magnetic field results in an osc. of the power absorption.



Lockin meas. the first derivative of the power absorption signal.